INTRODUCTION TO MATLAB FOR SOLVING AN ORDINARY DIFFERENTIAL EQUATION WITH INITIAL VALUE PROBLEM

Mariani Dian¹, a) and Catharina Mara Apriani², b)

¹,² Department of Mathematics Education, Faculty of Teacher Training and Education, Sanata Dharma University, Mrican, Tromol Pos 29, Yogyakarta 55002, INDONESIA

a) marianidian6@gmail.com
b) catharinaama@gmail.com

Abstract

One of the courses that existed in the mathematics education’s lecture is the Numerical Method. The Numerical Method is a method of approach to finding approach solutions, and commonly used if an analytic problem problem is difficult to find. To introduce MATLAB to students, a learning activity plan based on problem-based learning method was developed. Stages in the learning activity plan include problem recognition, analytic solution search, numerical method introduction, manual numerical solution search, numerical solution search using computer help, including use of Microsoft Excel as the introduction and MATLAB, and the meaning of the solution. The result of this planning is an example of ordinary differential equations with the initial value problem sought by the solution using the stages in the plan. The conclusion of this research are MATLAB is introduced to students because its use is more effective when compared to Microsoft Excel or manually ways, and MATLAB can be collaborated with problem based learning to create a new design of learning.

Keywords: MATLAB, ordinary differential equation, initial value problems, problem-based learning.

Introduction

Technology can be a powerful tool for transforming learning. It can help affirm and advance relationship between educators and students, reinvent our approaches to learning and collaboration, shrink long-standing equity and accessibility gaps, and adapt learning experiences to meet the needs of all learners (U.S. Department of Education, 2017). There are so many product of technology that has been created with the purpose to make a better experiences in learning. One of these product is MATLAB. There are some research that explain about MATLAB, one of them is in a research from GAO Feng (2011) about application of MATLAB in mathematical analysis, that presents several cases of MATLAB applications in mathematical...
analysis and the conclusion of this research is MATLAB can be a significant tool in mathematical analysis. One of the subjects taught to students of mathematics education at bachelor level is numerical method. Learning in the numerical course teaches the students to solve a case using a variety of ways. Mathematical problems can be solved by numerical methods such as systems of linear equations, systems of differential equations, and other systems of equations. Not all mathematical problems can be easily resolved analytically. Numerical methods are used to find solutions to a problem, especially one that is difficult to solve using analytical methods. Numerical calculations can be done manually or with the help of computers. Numerical calculations often require precision so that computer programming help such as Microsoft Excel or MATLAB is required. Microsoft Excel is a fairly simple and easy-to-use application for calculation, but the use of Microsoft Excel become less effective if iteration is too much, not to mention if iterative schema is complicated. One other application that can help numerical calculation is MATLAB. In this paper we want to answer about the question why using MATLAB to solved the problem and we also want to know if MATLAB can be collaborated with problem based learning to make a new design of learning.

Theory

A. Ordinary Differential Equations with Initial Value

Ordinary differential equations (ODE) is an equation for a missing function (or function) in term of derivatives of those functions. Recall that the derivative of a function $y(t)$ is denoted either by $y'(t)$ or by $dy/dt$. Then the derivative $y'(t_0)$ gives the slope tangent line to the graph of the function $y(t)$ at the point $(t_0, y_0)$ (Devaney, 2011). The best-known differential equation (and essentially the first example of a differential equation) is Newton’s second law of motion. Drop an object from the rooftop. If $y$ measuresthe position of the
center of mass of the object, then we would like to know its position at time \( t \), that is, \( y(t) \). Newton’s law tells us that mass times acceleration is equal to the force on the object. So, if \( m \) is the mass, then we have \( my'' = F(y) \), where \( F \) is the force acting on the object when it is in position \( y(t) \). So we have a differential equation for \( y(t) \) (Devaney, 2011).

A differential equation with additional terms to the unknown function and its derivatives, all given to the same value for the free variables, is an initial value problem (Nugraha, 2011). For example, we look at the unlimited population growth model from biology. Suppose we have a species living in isolation (with no predators, no overcrowding, and no emigration) and we want to predict its population as a function of time. Call the population \( y(t) \). Our assumption is that the rate of growth of the population is directly proportional to the current population. This translates to ODE \( y' = ky \). Here \( k \) is constant (a parameter) that depends on which species we are considering. We wish to find the solution of an initial value problem, that is, a specific solution of the ODE that satisfies \( y(0) = y_0 \) where \( y_0 \) is given initial population.

B. **MATLAB**

**MATLAB** is an interactive system for doing numerical computations. **MATLAB** relieves us of a lot of the mundane tasks associated with solving problems numerically. This allows you to spend more time thinking, and encourages us to experiment. **MATLAB** makes use of highly respected algorithms and hence we can be confident about our results. Powerful operations can be performed using just one or two commands. We can build up our own set of functions for a particular application (Griffiths, 2015). **MATLAB** stands for **MATRIX LABORATORY** (Matrix Laboratory), which was first used by Dr. CleveMoler in New Mexico University United States to teach courses of linear algebra. The basic data unit is a
matrix without dimension restriction. In MATLAB, computing become extremely easy. Dr. Moler in 1984 launched the official version of the software, in the later editions, he also gradually added to the control system, system identification, signal processing and communications, more than a toolbox, so MATLAB became to be widely used in automical control, image signal processing, biomedical engineering, signal analysis, optimization and other fields (Feng, 2011).

In pure mathematics, since MATLAB is an integrated computer software which has three functions: symbolic computing, numerical computing and graphic drawing, MATLAB is capable to carry out many functions including computing polynomials and rational polynomials, solving equations and computing many kinds of mathematical expression. One can also use MATLAB to calculate the limit, derivative, integral and Taylor Series of some mathematical expressions. With MATLAB, the graph of functions with one or two variables can be easily drawn in selected domain (Feng, 2011).

C. Problem Based Learning

Problem-based learning has often been understood simply as a method of learning. Correspondingly, many kinds of pragmatically based pedagogical applications and development project are described as PBL. Problem-based learning has also been investigated within the context of education, although the theoretical basis of problem-based learning is closely connected with learning at work (Poikela, 1998; Karila & Nummenmaa, 2001; Poikela & Jarvinen, 2001; Poikela & Poikela, 2001 in Poikela & Nummenma, 2006). Problem-based learning as an educational methodology started at the McMaster University, Canada in 1969. Problem-based on actual clinical cases were used as focal points in a medical program (Ee, 2009). The problem-based learning process essentially consists of the
following stage: (1) meeting the problem; (2) problem analysis and generation of learning issues; (3) discovery and reporting; (4) Solution presentation and reflection; and (5) overview, integration, and evaluation, with self-directed learning bridging one stage and the next (Tan, 2003 in Tan, 2009).

Methods (Design for The Learning Activity)

Broadly speaking, the activity design composed includes 6 stages, which are:

1. Problem Introduction

   The problem recognition process aims to provide a context of problems to students, so that their purpose of learning a method becomes clearer. In the process of introducing this problem, students are expected to know what is really a problem in a case. At this stage students analyze the problem, then determine a hypothesis related problems.

2. Search for Analytic Solutions

   This stage is the stage where students are invited to find a suitable strategy to solve the problem analytically. Students are then expected to apply the strategies they have to find an analytical solution.

3. Introduction of the Approach Methods

   The next stage is the introduction of approach methods. At this stage students are introduced to alternative methods that can also be used to find solutions to the problems. Students then analyze the methods until finally making a plan to find a solution of the problem. The draft strategy is generally an iterative scheme.

4. Search for The Approach Solutions Manually
In the next stage students are expected to apply strategies they have designed manually. The goal is to make the students really understand the iterative schemes they have previously obtained.

5. Search for The Solutions Using Computer Help

This stage involves searching the approach solutions using *Microsoft Excel* and *MATLAB* applications. In using the approach method, generally iteration required quite a lot of number. When the required iteration and iterative scheme used are quite complicated, then the process of finding solutions will take quite a long time and requires considerable accuracy. Therefore, in order for the process of finding solutions to work more effectively, computer assistance is required.

First, students are expected to use Microsoft Excel to find an approach solution, by applying iterative schema into the application. To compare, an analytic solution scheme (exact solution) is provided, so we can look for errors at each point (the corresponding per-cell error).

The next stage, students are introduced to MATLAB, because in some cases, often the use of Microsoft Excel is considered less effective. To use MATLAB students must have already understood the basic algorithms in programming. Since this process involves an iterative schema, the algorithm that must be mastered is a repetition algorithm.

6. The Meaning of Solutions

In the last stage of the learning activities, students are expected to review the problems that have been previously given. Students are then asked to link the solutions that have been obtained with the context of the problem, to see whether the results match or not with the hypothesis that has been proposed at the beginning of the learning process.
Result and Discussion (An Example of ODE with Initial Value Problem)

A. Problem Introduction

An object that made from metal, 0.5 kg heavy and the initial temperature is 300°C (573 K) dipped in water that have temperature at 25°C (298 K), where there is a natural conventionally cooling process occurs.

\[
\frac{dT}{dt} = \frac{A}{ρcv} h_c(298 - T)
\]  

(1)

In the equation (1), the temperature is in K (Kelvin) unit, with the constants:

density, \( ρ = 300 \) kg/m³

e volume , \( v = 0.005 \) m³

surface area, \( A = 0.25 \) m²

Heat, \( c = 900 \) J/KgK

Heat transfer coefficient, \( h_c = 300 \) W/m²K

From the problem, we will look for the temperature plot of the object for 5 minutes (quoted from Kosasih, 2006).

Hypothesis: Logically, when a metal cools, then the plot temperature for 5 minutes will decrease from the initial temperature to a certain temperature.

B. Analytic Solutions

From the given problem, then we can obtain the analytic solution from that problem like this:

From the given problem we obtaining a system for an ordinary differential equation (ODE) with initial value problem:

\[
\left\{ \begin{array}{l}
\frac{dT}{dt} = \frac{A}{ρcv} h_c(298 - T) \\
T(0) = 573
\end{array} \right.
\]  

(2)
For \( 0 \leq t \leq 5 \).

Where:

\( t = \text{time (in minute)} \)

\( T = \text{temperature} \)

Analytical solution for the equation system (2) and (3) as follows:

\[
\frac{dT}{(298-T)} = \frac{A}{\rho cv} h_c dt
\]

(4.a)

\[
\int \frac{dT}{(298-T)} = \frac{A}{\rho cv} h_c \int dt
\]

(4.b)

\[
-ln |(T - 298)| = \frac{A}{\rho cv} h_c t + C
\]

(4.c)

\[
T = 298 + e^{-\frac{A}{\rho cv} h_c t}. e^C
\]

(4.d)

From equation (3) was known that \( T(0) = 573 \), then \( e^C \) can be found by this information:

\[
573 = 298 + e^{-\frac{A}{\rho cv} h_c 0}. e^C
\]

(5.a)

So that:

\[
e^C = 275
\]

(5.b)

By substituting (5.b) to (4.d) we get the analytic solution:

\[
T = 298 + 275e^{-\frac{A}{\rho cv} h_c t}
\]

(6)

C. Introduction of the Approach Methods

At this stage, using the problem in equation (1), two approach methods are considered suitable to find the solution, that are Euler Method and Heun Method. Students are invited to actively communicate the two methods. One of the alternatives for the introduction of these two methods of approach is counselors with some discussion with the students describe the Taylor Series to get an iterative scheme of Euler Method for the solution completion. After that students are asked to discuss the iterative scheme obtained for Heun Methods based on
the information they have gained about the Euler Method and the idea of Method Heun. Here is one form of Euler Method description and Heun Method (Aminuddin, 2006).

1. Euler Method

Euler Method is a method for solving the ODE using the Taylor series.

\[ y' = \frac{dy}{dx} = f(x,y) \]  

(7)

The iterative scheme for Euler Method is:

\[ y_{i+1} = y_i + \Delta x f(x,y) \]  

(8)

From the equation (8) can be concluded that the scheme of the solution for the previous problem, using the Euler Method is:

\[ T_{(i+1)} = T_i + hf(t_i, T_i) \]  

(9)

Note that \( h = \Delta t \), for \( i = 0, 1, 2, 3, ... \)

2. Heun Method

This method estimates two gradients at intervals, that is at the beginning and end points. The best value is obtained from averaging between the gradient at the beginning and end points. The gradient at the earliest end of the interval obtained by the Euler Method is expressed in the form of equation (7). Then linear extrapolation is applied to the value of \( y_{i+1} \):

\[ y_{i+1}' = y_i + \Delta x f(x,y) \]  

(10)

\[ y_{i+1} = y_i + \frac{1}{2} \Delta x (f(x_i, y_i) + f(x_{i+1}, y_{i+1}')) \]  

(13)

From the equation (10) and (13), obtained the scheme of resolving the initial value problem that already described above using Heun method, that is:
D. The Approach Solutions Manually

Consider $h = 1$, which means that the interval of time for predicting the temperature every minute. So that, in the 5 minutes, we need 5 numbers of iteration to get the temperature at the 1st, 2nd, 3rd, 4th, and 5th minute (in this stage we are not solve the all of the problem yet, because the problem ask the plot for 5 minutes). Here is one form of solution using the Euler Method and The Heun Method.

1. Using Euler Method

Using the iterative scheme (9), we get the solutions for $h = 1$ are:

$$T_i = T_0 + hf(t_0, T_0)$$

$$T_1 = 573 + \frac{3500 \times 0.25}{900 \times 0.005} \times (298 - 573) = 394.7593 \text{ (The object temperature at the first minute is } 394.7593 \text{ K)} }$$

$$T_2 = T_1 + hf(t_1, T_1)$$

$$T_2 = 557.72 + \frac{3500 \times 0.25}{900 \times 0.005 \times 0.300} \times (298 - 394.7593) = 332.0449 \text{ (The object temperature at the second minute is } 332.0449 \text{ K) }$$

The iteration process is done until the 5th minute.

2. Using Heun Method

Using the iterative schemes (11.a) and (11.b), we obtained the solutions for $h = 1$, which are:

$$T_{i+1} = T_i + hf(t_i, T_i)$$

$$T_{i+1} = T_i + \frac{1}{2} h[f(t_i, T_i) + f(t_{i+1}, T_{i+1})]$$
The iteration process is done until the 5th minute.

E. The Approach Solutions Using Computer Help

There is some calculation results using Microsoft Excel for Euler Method and Heun Method, and the comparison of error from both of that method.

<table>
<thead>
<tr>
<th>N</th>
<th>t(i)</th>
<th>Euler</th>
<th>Analytic</th>
<th>Error</th>
<th>t(i)</th>
<th>T*</th>
<th>Heun</th>
<th>Analytic</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>573</td>
<td>573</td>
<td>0</td>
<td>0</td>
<td>573</td>
<td>573</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>394,7593</td>
<td>441,8287</td>
<td>47,0694</td>
<td>1</td>
<td>394,7593</td>
<td>452,5225</td>
<td>441,8287</td>
<td>10,6938</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>332,0449</td>
<td>373,2243</td>
<td>41,1794</td>
<td>2</td>
<td>352,3690</td>
<td>384,8262</td>
<td>373,2243</td>
<td>11,6018</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>309,9788</td>
<td>337,3433</td>
<td>27,3646</td>
<td>3</td>
<td>328,5499</td>
<td>346,7876</td>
<td>337,3433</td>
<td>9,4443</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>302,2148</td>
<td>318,5771</td>
<td>16,3623</td>
<td>4</td>
<td>315,1660</td>
<td>325,4137</td>
<td>318,5771</td>
<td>6,8367</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>299,4830</td>
<td>308,7621</td>
<td>9,2791</td>
<td>5</td>
<td>307,6456</td>
<td>313,4038</td>
<td>308,7621</td>
<td>4,6417</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N</th>
<th>t(i)</th>
<th>Euler</th>
<th>Analytic</th>
<th>Error</th>
<th>t(i)</th>
<th>T*</th>
<th>Heun</th>
<th>Analytic</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>573</td>
<td>573</td>
<td>0</td>
<td>0</td>
<td>573</td>
<td>573</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0,2</td>
<td>528,439</td>
<td>531,862</td>
<td>3,422</td>
<td>0,2</td>
<td>528,439</td>
<td>532,050</td>
<td>531,862</td>
<td>0,187</td>
</tr>
</tbody>
</table>
From both tables above obtained that the approach solution using Euler Method and Heun Method for $h = 1$ and $h = 0.25$. From that tables we can see that for the smaller value of $h$ then the approach solutions become more accurate to the analytic solutions. The
temperatures plot for the metal object can be created using Microsoft Excel or using MATLAB (in this case we will using the MATLAB’s programs to make the temperatures plot for the problem)

This is one of the examples for the programs which can be made by the students using MATLAB application.

1. MATLAB’s program for the function

   ```matlab
   function z=fk(t,T) % function on the right hand of ODE
   z=0.25*3500*(298-T)/(300*900*0.005); % Equation
   end
   ```

2. MATLAB’s program for the Euler Method

   ```matlab
   clear all;
   clc
   a=0; % initial time
   b=5; % the end time (in minutes)
   h=1/60 % time interval (delta t)
   t=a:h:b;
   N=length(t); % the number of points t
   for i=2:N
       t(1)=0; % initial value of t
       T(1)=573; % initial value of T
       T(i)=T(i-1)+h*fk(t(i-1),T(i-1)); % Euler Method scheme
   end
   TE= 298+275*exp((-((0.25*3500)/(300*900*0.005))*t)); % Analytic T
   plot(t,TE,'k', t,T,'b-') % solution graph
   legend('TEksak','TNumeris')
   xlabel('t')
   ylabel('T')
   Error=(1/N)*sum(abs(TE-T)) % error calculation
   tabel =[T' TE']; % making a table
   disp(tabel) % display the table
   ```

3. MATLAB’s Program for Heun Method

   ```matlab
   clear all;
   clc
   a=0; % initial time
   b=5; % the end time (in minute)
   h=1/60 % time interval (delta t)
   t=a:h:b;
   ```
N=length(t); % the number of points t
for i=2:N
    t(1)=0; % initial value of t
    T(1)=573; % initial value of T
    % Scheme of Heun Method
    Tib=T(i-1)+h*fk(t(i-1),T(i-1)); % T(i) star
    T(i)=T(i-1)+0.5*h*[fk(t(i-1),T(i-1))+fk(t(i),Tib)];
end
TE= 298+275*exp(-((0.25*3500)/(300*900*0.005))*t); % Analytic T
plot(t,TE,'k', t,T,'b-') % Solution graph
legend('TEksak','TNumeris')
xlabel('t')
ylabel('T')
Error=(1/N)*sum(abs(TE-T)) % error calculation
tabel = [T' TE']; % making a table
disp(tabel) % display the table

We can change the function depends on the problem. Other than that, in the program for
Euler and Heun Methods, the a, b and h variables can be customized with the given problem.

F. The Meaning of Solutions

1. Euler Method

Using the example of the program in the previous stage, we obtain the solution graph like
this.

Figure 1. Euler’s solution graph for

\[ h = 0.25 \]

Figure 2. Euler’s solution graph for

\[ h = \frac{1}{60} \]
By looking at Fig. 1. and Fig. 2. we can see that a smaller $h$ value will give us more accurate approximation solutions (using the program obtained the total error for $h = 0.25$ is 5.7844 while the total error for $h = \frac{1}{60}$ is 0.3825).

2. Heun Method

Using the example of the program in the previous stage, we obtain the solution graph like this.

![Figure 3](image1.png) Heun’s solution graph for $h = 0.25$

![Figure 4](image2.png) Heun’s solution graph for $h = \frac{1}{60}$

From Fig. 3. and Fig. 4. we can see that the two approach graphs are quite accurate when compared to the previous method for each of the same $h$ values (from the program the total error is 0.3375 for $h = 0.25$, while the total error for $h = \frac{1}{60}$ is 0.0014). Overall, we can see from the solution graph (both in Euler Method and Heun Method) that the metal temperature decreased from the initial temperature of 573 K to close to 300 K within 5 minutes (from the temperature calculation after 5 min about 308 K). In addition we can see that in this case, the use of Method Heun is more effective than the use of Euler Method, since the error obtained from Method Heun is less than the error obtained from the Euler Method, for the same time interval.
Conclusion

MATLAB is introduced to students because its use is more effective when compared to manual or using Microsoft Excel, especially when iterations are needed in large numbers and iterative schemes are quite complicated. The use of MATLAB can minimize errors caused by inaccuracy in calculations. To use MATLAB students are required to master basic programming algorithms, especially repetition algorithms, and fully understand the iterative scheme of the approach method to be used. Also, we can see that MATLAB can be collaborated with problem based learning to create a new method of learning.

References


